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A System with Three Limit Cycles Appearing in a Hopf Bifurcation and Dying in a Homoclinic Bifurcation: The Cusp of Order 4*

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In this paper we give an example of a family of polynomial vector fields with three limit cycles appearing simultaneously on a Hopf bifurcation (H) of order 3 and vanishing simultaneously in a homoclinic loop bifurcation (HL) of order 3. The region with three limit cycles is a topological 3-simplex. The system is a generalization of Bogdanov's system. At the same time we give the bifurcation diagram for the universal unfolding of the cusp of order 4. This bifurcation diagram is a cone. It contains a cone on the bifurcation diagram with the three limit cycles inside a 3-simplex region, plus saddle-node and cusp bifurcations of lower order. © 1989 Academic Press, Inc.

1. INTRODUCTION

In 1975, Bogdanov [4] studied the following system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -1 + x^2 + \mu_1 y + \mu_2 xy, \end{aligned} \tag{1.1}$$

with $\mu_2 \neq 0$. He observed the birth of a limit cycle in the system from the sinular point $(-1, 0)$, its monotonic growth, and its disappearance when it reached the saddle point $(1, 0)$. In this study Bogdanov considered the system as depending on the essential parameter $v = \mu_1/\mu_2$.

Bogdanov then applied this result to the study of bifurcations of a

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singular point of a vector field [5]. There he was concerned with the codimension 2 singularity corresponding to a Jordan block with a pair of zero eigenvalues.

Such a singularity has normal form [7]:

$$\begin{aligned}\dot{x} &= y + O(|x, y|^3)\\ \dot{y} &= ax^2 + bxy + O(|x, y|^3).\end{aligned} \tag{1.2}$$

A singularity with normal form (1.2) satisfying $a \neq 0$ is called a *cusp* (from the geometric shape of separatrices passing through the origin). In the codimension 2 case we have $b \neq 0$. This singularity (called a *Bogdanov–Takens bifurcation* [4, 5, 16]) has universal unfolding (under rescaling):

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= \varepsilon_1 + \varepsilon_2 y + x^2 \pm xy.\end{aligned} \tag{1.3}$$

The first complete proof of the universality of the unfolding is due to Bogdanov [4, 5], where he uses a change of variables bringing (1.3) to (1.1) in the case where the system has two different singular points, i.e., $\varepsilon_1 < 0$. System (1.1) is then analysed by Melnikov's method as a perturbation of the Hamiltonian system,

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -1 + x^2,\end{aligned} \tag{1.4}$$

with the Hamiltonian function:

$$H(x, y) = y^2/2 + x - x^3/3. \tag{1.5}$$

In the bifurcation diagram, $\nu = \mu_1/\mu_2$ is the essential parameter, and the bifurcation diagram is a cone built on a one-parameter bifurcation diagram in ν-space.

Later, in [8], Dumortier *et al.* studied what they called the *cusp of order* 3. They considered system (1.2), in the case $b = 0$, but had to place a non-degeneracy condition on the higher order terms in order to achieve a codimension 3 bifurcation. Accordingly they analysed the system:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= \varepsilon_1 + \varepsilon_2 y + x^2 + \varepsilon_3 xy + x^3 y,\end{aligned} \tag{1.6}$$

with ε_1, ε_2, ε_3 small parameters, through a change of variables which, for $\varepsilon_1 < 0$, brought the system to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -1 + x^2 + \mu_1 y + \mu_2 xy + \mu_3 x^3 y, \end{aligned} \tag{1.7}$$

with $\mu_3 > 0$. A change of parameters yields here two essential parameters:

$$\nu_1 = \mu_1/\mu_3 \qquad \nu_2 = \mu_2/\mu_3. \tag{1.8}$$

The bifurcation diagram of this system is a cone on the bifurcation diagam in ν-space. In this bifurcation diagram we find a triangular region (a "topological two simplex") in which the system has two limit cycles (Fig. 1).

The reason why the x^2y-term is not considered in (1.6) is well explained in [3, 8, 13]. In [8, 13] it is shown that the terms $x^{3k+2}y$ in the second equation play no role in the bifurcation diagram, while in [3] it is shown how these terms disappear as a result of further normal form reduction.

The number of limit cycles there are for given values of ν_1 and ν_2 can also be determined geometrically. Limit cycles arising from a fixed level curve $H = h$ of (1.5) occur on a line in the ν-plane, which is tangent to the curve (2C) (Fig. 2). For a given point (ν_1, ν_2) the number of limit cycles is equal to the number of tangents to the curve (2C) that can be drawn through the point.

The examples of Eqs. (1.1) and (1.7) illustrate the duality between Hopf and homoclinic loop bifurcation of order 1 and 2, as described by Joyal in

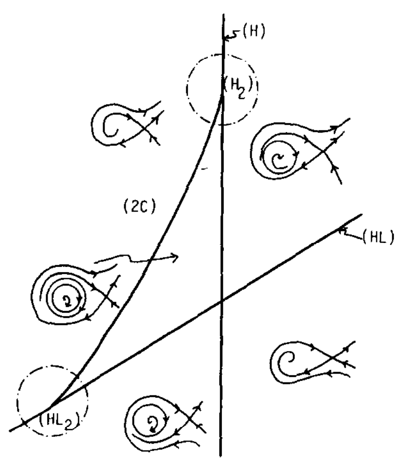

FIG. 1. Bifurcation diagram for a family of vector fields with two cycles appearing in (H_2) and dying in (HL_2). This shows duality between (H_2) and (HL_2).

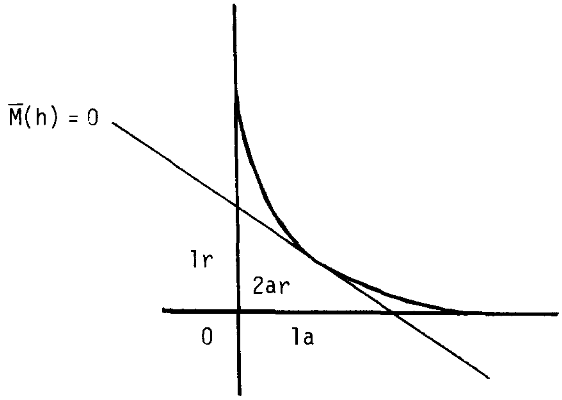

FIG. 2. Lines $\bar{M}(h)=0$. In each open region in the bifurcation diagram we give the number of limit cycles: The indices indicate their stability from the inner to the outer.

[10]. The bifurcation diagrams have the same topological shape. In Hopf bifurcation of order 1 (resp. homoclinic loop bifurcation of order 1), the topological type of the vector field stays the same at large scale (resp. far from the homoclinic loop), but changes at small scale (resp. close to the loop). Therefore, a limit cycle is needed to glue together the two behaviours. Similarly, for Hopf bifurcation of order 2 and homoclinic bifurcation of order 2, the topological organization of the regions with zero, one, and two limit cycles is identical in the neighborhoods of the bifurcations.

In this paper we consider (1.7) with an additional degeneracy. We prove that this yields a bifurcation diagram in which the region with three limit cycles is a *topological three simplex.* More precisely, we study the system,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -1 + x^2 + \mu_1 y + \mu_2 xy + \mu_3 x^3 y + \mu_4 x^4 y, \end{aligned} \tag{1.9}$$

under the restriction $\mu_4 \neq 0$, where the parameters μ_i are small. As in the previous cases, we change to essential parameters,

$$\mu_1 = \nu_1 \mu_4, \qquad \mu_2 = \nu_2 \mu_4, \qquad \mu_3 = \nu_3 \mu_4, \tag{1.10}$$

and we get a bifurcation diagram in ν-space which is independent of the value of μ_4 (Fig. 3). System (1.9) is studied as a perturbation of the Hamiltonian system (1.4), and limit cycles are considered as arising from closed level curves $H = h$, with H given in (1.5).

We also give a geometric construction of the bifurcation diagram and

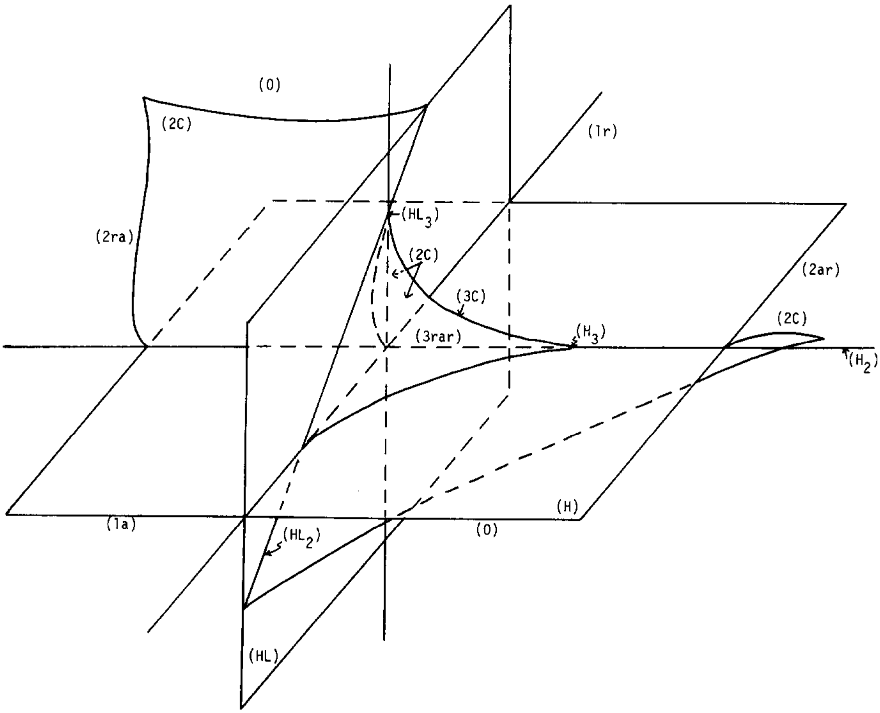

FIG. 3. Bifurcation diagram for a family of vector fields with three limit cycles appearing in a (H_3) and dying in a (HL_3). This shows duality between (H_3) and (HL_3).

determine the number of limit cycles for each point in ν-parameter space geometrically.

This example shows that the duality persists between Hopf bifurcation of order 3 (H_3) and homoclinic loop bifurcation of order 3 (HL_3). We can see from Fig. 3 that the bifurcation diagrams in the neighborhood of the two singularities are the same.

In the second part of the paper, we show how these results can be used as part of the proof that

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon_1 + \varepsilon_2 y + x^2 + \varepsilon_3 xy + \varepsilon_4 x^3 y \pm x^4 y \end{aligned} \tag{1.11}$$

is a universal unfolding of

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 \pm x^4 y. \end{aligned} \tag{1.12}$$

The proof uses a "blowing up," as in the cases of Bogdanov and the cusp of order 3. We find in particular that the maximum number of limit cycles in that case is three, a result compatible with the conjecture in [13]: "The maximum number of cycles appearing in the universal unfolding of a cusp of codimension k is $(k-1)$."

In Section 2 we derive the bifurcation diagram of system (1.9). Finally, in Section 3 we apply the previous results to the cusp of order 4.

2. Analysis of System (1.9)

We restrict ourselves to the case $\mu_4 > 0$. The case $\mu_4 < 0$ can be obtained by a change of variables $y \mapsto -y$, $t \mapsto -t$. Throughout this section it is assumed that the μ_i are small parameters. In fact, as we will see, it is enough to suppose that the essential parameters ν given in (1.10) are inside a compact set, and that the parameter μ_4 is sufficiently small.

In the first subsection we study Hopf bifurcation. In the second subsection we briefly explain Melnikov's method, and in the third subsection we apply this method to the study of homoclinic loop bifurcation. In Subsection 2.4 we give Riccati's equation, and use it to deduce the fundamental theorem: System (1.9) has at most three limit cycles. We apply Riccati's equation, together with Melnikov's method, to the study of multiple limit cycles in Subsection 2.5. Finally in Subsection 2.6 we describe the geometrical shape of the bifurcation diagram.

Notation throughout the Paper

— (H) (resp. (H_2), (H_3)) denotes Hopf bifurcation of order 1 (resp. 2, 3).

— (HL)(resp. (HL_2), (HL_3)) denotes homoclinic loop bifurcation of order 1 (resp. 2, 3).

— (H, HL) denotes simultaneous occurrence of (H) and (HL), similarly for (H_2, HL) and (H, HL_2).

— (2C) denotes bifurcation of a double limit cycle.

— (3C) denotes bifurcation of a triple limit cycle.

— (BT) or (C_2) denotes Bogdanov–Takens bifurcation (cusp of order 2). Similarly (C_3) denotes the cusp of order 3.

— (H, 2C) denotes simultaneous occurrence of (H) and (2C), and similarly for (HL, 2C).

— (SN) denotes saddle-node bifurcation.

System (1.9) has singular points $(-1, 0)$ and $(1, 0)$. The point $(1, 0)$ is always a saddle. The point $(-1, 0)$ is either a focus or a weak focus.

2.1. *Hopf Bifurcation at the Point* $(-1, 0)$

We linearize the system at $(-1, 0)$ by means of a change of variable $X = x + 1$:

$$\begin{aligned} \dot{X} &= y \\ \dot{y} &= -2X + y(\mu_1 - \mu_2 - \mu_3 + \mu_4) + X^2 + (\mu_2 + 3\mu_3 - 4\mu_4)\, Xy \\ &\quad + (-3\mu_3 + 6\mu_4)\, X^2 y + (\mu_3 - 4\mu_4)\, X^3 y + \mu_4 X^4 y. \end{aligned} \tag{2.1.1}$$

The point is stable (unstable) if $\mathrm{tr} = \mu_1 - \mu_2 - \mu_3 + \mu_4 < 0\ (>0)$. It undergoes a Hopf bifurcation when

$$\mathrm{tr} = \mu_1 - \mu_2 - \mu_3 + \mu_4 = 0. \tag{2.1.2}$$

To study Hopf bifurcation at (2.1.2), we make the change of coordinates:

$y = -\sqrt{2}\, Y$. Equation (2.1.1) becomes

$$\begin{aligned} \dot{X} &= -\sqrt{2}\, Y \\ \dot{Y} &= \sqrt{2} X - X^2/\sqrt{2} + (\mu_2 + 3\mu_3 - 4\mu_4)\, XY \\ &\quad + (-3\mu_3 + 6\mu_4)\, X^2 Y + (\mu_3 - 4\mu_4)\, X^3 Y + \mu_4 X^4 Y. \end{aligned} \tag{2.1.3}$$

The order of Hopf bifurcation is given by the index of the first non-zero Lyapunov coefficient (see [15] and Appendix). The first Lyapunov coefficient is given by

$$V_1 = \mu_2 - 4\mu_3 + 8\mu_4. \tag{2.1.4}$$

When $V_1 = 0$, the system depends on μ_3 and μ_4 alone. We get the following values for the second and third Lyapunov coefficients:

$$V_2 = (5\mu_3 - 14\mu_4)/96\sqrt{2}; \tag{2.1.5}$$

when $V_2 = 0$, we get for V_3:

$$V_3 = 14\mu_4/5 > 0. \tag{2.1.6}$$

Hopf bifurcation of order 3 therefore takes place at

$$\mu_1 = 11\mu_4/5, \qquad \mu_2 = 2\mu_4/5, \qquad \mu_3 = 14\mu_4/5. \tag{2.1.7}$$

i.e.,

$$(\nu_1, \nu_2, \nu_3) = (11/5,\ 2/5,\ 14/5) \tag{2.1.8}$$

using the essential parameters defined in (1.10). There are three limit cycles inside the region $V_2<0$, $V_1>0$, $\mathrm{tr}<0$, i.e.,

$$\begin{aligned}
&5v_3-14<0,\\
&v_2-3v_3+8>0,\\
&v_1-v_2-v_3+1<0,\\
&\text{with } |v_1-v_2-v_3+1| \ll |V_1| \ll |V_2| \ll |V_3|.
\end{aligned} \tag{2.1.9}$$

2.2. *Melnikov's Method*

This method is used to approximate the parameter values for which a perturbation of a Hamiltonian system has a homoclinic loop. It is more general since, for each closed loop of the Hamiltonian system, it gives the parameter values for which this closed loop is preserved in the perturbed system.

We consider a system of the form:

$$\begin{aligned}
\dot{x} &= \partial H/\partial y + \varepsilon f(x, y) = F(x, y)\\
\dot{y} &= -\partial H/\partial x + \varepsilon g(x, y) = G(x, y),
\end{aligned} \tag{2.2.1}$$

and the level curves $H=h$ of the Hamiltonian system, called γ_h. We want to know which closed curves γ_h of the unperturbed system remain closed trajectories γ'_h after perturbation. Accordingly we must have

$$\begin{aligned}
N(h) &= \int_{\gamma'_h} dH = \int_{\gamma'_h} \dot{H}\, dt = \varepsilon \int_{\gamma'_h} (f\partial H/\partial x + g\partial H/\partial y)\, dt = 0\\
&= \varepsilon \int_{\gamma_h} (f\,\partial H/\partial x + g\,\partial H/\partial y)\, dt + o(\varepsilon) = \varepsilon \int_{\gamma_h} (g\,dx - f\,dy) + o(\varepsilon),
\end{aligned} \tag{2.2.2}$$

since γ'_h is close to γ_h. The zeros of the function $N(h)$ are well approximated by the zeros of Melnikov's function:

$$M(h) = \int_{\gamma_h} (g\,dx - f\,dy). \tag{2.2.3}$$

In the case where γ_h is an ordinary closed curve of the Hamiltonian, if $M(h)=M'(h)=\cdots=M^{(k-1)}(h)=0$, and $M^{(k)}(h)\neq 0$, then a cycle of multiplicity k appears in the perturbation.

In the case where γ_h is a homoclinic loop [14], we have a homoclinic loop bifurcation of order:

(i) $2k-1$, if $M(h)=M'(h)=\cdots=M^{(k-1)}(h)=0$ and $M^{(k)}(h)=\pm\infty$,

(ii) $2k$, if $M(h) = M'(h) = \cdots = M^{(k-1)}(h) = 0$ and $M^{(k)}(h) \neq 0$ with $M^{(k)}(h)$ finite.

Note that $M'(h)$ is finite if and only if the trace of the saddle point is zero. Under the conditions $M(h) = M'(h) = \cdots = M^{(k-1)}(h) = 0$, we can decide whether $M^{(k)}$ is finite or not by looking at the *Poincaré normal form* of the perturbed vector field at the saddle point [2]. This normal form is given by

$$\begin{aligned} \dot{x} &= x + c_1 x^2 y + c_2 x^3 y^2 + \cdots + c_k x^{k+1} y^k + O(|x, y|^{2k+2}) \\ \dot{y} &= y + d_1 x y^2 + d_2 x^2 y^3 + \cdots + d_k x^k y^{k+1} + O(|x, y|^{2k+2}). \end{aligned} \tag{2.2.4}$$

The *saddle quantities* are defined as $u_i = c_i + d_i$. $M(h) = M'(h) = \cdots = M^{(k-1)}(h) = 0$ implies (but is not equivalent to) the fact that the first $(k-2)$ saddle quantities are zero. Then $M^{(k)}(h)$ is finite if and only if $u_{k-1} = 0$. Consequently the saddle quantities control the finiteness of the $M^{(i)}(h)$ [10, 12].

2.3. *Determination of Homoclinic Loop Bifurcation*

We now return to the discussion of (1.9). The set in parameter space corresponding to a homoclinic loop is well approximated by the zeros of

$$M(2/3) = \int_\gamma (v_1 y + v_2 x y + v_3 x^3 y + x^4 y)\, dx = 0, \tag{2.3.1}$$

where $M(h)$ is the Melnikov function,

$$M(h) = \int_{\gamma_h} (v_1 y + v_2 x y + v_3 x^3 y + x^4 y) dx = 0, \tag{2.3.2}$$

γ_h is the level curve $H = h$ of the Hamiltonian function (1.5) (Fig. 4), and γ is the homoclinic loop corresponding to $h = 2/3$. The parameter ε of Subsection 2.2 is given by μ_4, and the parameters v are given in (1.10). They assume values inside a compact region of v-space. On $H = 2/3$, we have

$$y = \sqrt{2/3}(1 - x)\sqrt{x + 2}, \tag{2.3.3}$$

resulting in

$$M(2/3) = 24\sqrt{2}(v_1 - 5v_2/7 - 103v_3/77 + 187/91)/5 = 0. \tag{2.3.4}$$

Homoclinic loop bifurcations of higher order can be determined by calculating $M'(2/3)$. If $M'(2/3)$ is infinite (resp. finite, zero), we have a homoclinic loop bifurcation of order 1 (resp. 2, 3, or higher),

$$M'(2/3) = \sqrt{3}/2 \int_\gamma (v_1 + v_2 x + v_3 x^3 + x^4)/y\, dx. \tag{2.3.5}$$

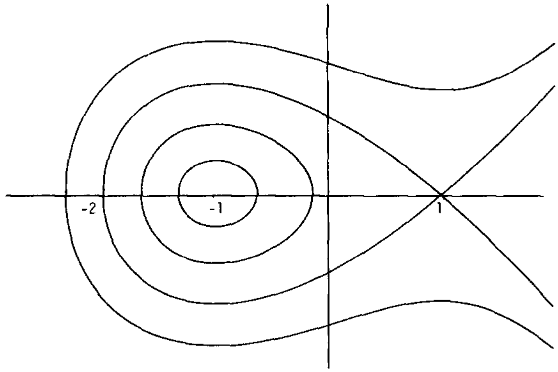

FIG. 4. Level curves of (1.5).

A direct calculation, replacing y by its value in (2.3.3), shows that $M'(2/3)$ is finite if and only if the trace of the saddle point is zero:

$$\text{tr} = v_1 + v_2 + v_3 + 1 = 0. \tag{2.3.6}$$

According to Hypotheses (2.3.4) and (2.3.6) we get

$$M'(2/3) = 6(-v_2 - 9v_3/5 + 8/7). \tag{2.3.7}$$

The first saddle quantity for system (1.9) is given by [11]

$$u_1 = (-v_2 + 3v_3 + 8v_4)/2. \tag{2.3.8}$$

Hypotheses (2.3.4), (2.3.6), and $M'(2/3) = 0$ lead to $u_1 > 0$.

PROPOSITION 2.3.1. *For sufficiently small μ_i, homoclinic loop bifurcation of order* 3 (HL_3) *takes place at*

$$(v_1, v_2, v_3) = (-107/91, \ -94/91, \ 110/91), \tag{2.3.9}$$

and the three limit cycles appear in the region

$$\begin{aligned} -v_2 - 9v_3/5 + 8/7 &< 0, \\ v_1 + v_2 + v_3 + 1 &> 0, \\ v_1 - 5v_2/7 - 103v_3/77 + 187/91 &> 0. \end{aligned} \tag{2.3.10}$$

Proof. The first statement follows from the previous calculations. $M(h)$ represents approximately the variation of h along the trajectory $M(h) = 1/\varepsilon \int \dot{H}\, dt$. The homoclinic loop lies at the maximum value of h. If we make the change of variables $\bar{h} = 2/3 - h$, then we can consider

$\tilde{M}(\bar{h}) = 1/\varepsilon \int \dot{\bar{h}}\, dt = -M(h)$. $\tilde{M}(0) = 0$ at the homoclinic loop. For $\tilde{M}(0) < 0$ (resp. > 0), the stable manifold W^s is inside (resp. outside) the region surrounded by the unstable manifold W^u. We start with $u_1 > 0$ and $\tilde{M}(0) = \tilde{M}'(0) = 0$, i.e., with a repulsive homoclinic loop. We add a small perturbation in order to get $-\infty < \tilde{M}'(0) < 0$ inside $\tilde{M}(0) = 0$. This adds a first limit cycle. The second, much smaller perturbation brings $\tilde{M}'(0)$ to $+\infty$ by taking a positive value for the trace [1], giving a second limit cycle. We then break the repulsive homoclinic loop, with an even smaller perturbation and obtain the third limit cycle when W^s is inside the region surrounded by W^u. i.e., when $\tilde{M}(0) < 0$, which is the same as $M(2/3) > 0$. Noting that $d\tilde{M}(\bar{h})/d\bar{h}\,|_{\bar{h}=0} = dM(h)/dh\,|_{h=2/3}$ completes the proof. Conditions (2.3.10) are consistent since they are verified at Hopf bifurcation of order 3. ∎

Similarly, homoclinic loop bifurcation of order 2 occurs together with Hopf bifurcation of order 1(H, HL_2) at

$$(v_1, v_2, v_3) = (-1, -22/13, 22/13), \tag{2.3.11}$$

and the three limit cycles appear in the region

$$\begin{aligned} v_1 - v_2 - v_3 + 1 &< 0, \\ v_1 + v_2 + v_3 + 1 &> 0, \\ v_1 - 5v_2/7 - 103v_3/77 + 187/91 &> 0. \end{aligned} \tag{2.3.12}$$

Likewise homoclinic loop bifurcation of order 1 occurs together with Hopf bifurcation of order 2(H_2, HL) at

$$(v_1, v_2, v_3) = (31/65, -58/65, 154/65), \tag{2.3.13}$$

and the three limit cycles appear in the region

$$\begin{aligned} v_2 - 3v_3 + 8 &> 0, \\ v_1 - v_2 - v_3 + 1 &< 0, \\ v_1 - 5v_2/7 - 103v_3/77 + 187/91 &> 0. \end{aligned} \tag{2.3.14}$$

Remark. In each case ((2.1.9), (2.3.10), (2.3.12), and (2.3.14)), the equalities are consistent. The intersection of the four regions is non-empty.

2.4. *Riccati's Equation*

System (1.9) has a limit cycle for small μ, collapsing to γ_h for $\mu = 0$ if the function $M(h) = 0$. The limit cycle is unique if $M'(h) \neq 0$. Otherwise we will have a multiple limit cycle. Since $M(h)$ is an elliptic integral we must introduce new tools to study it. Let

$$I_0=\int_{\gamma_h} y\,dx, \qquad I_1=\int_{\gamma_h} xy\,dx, ..., \qquad I_n=\int_{\gamma_h} x^n y\,dx. \tag{2.4.1}$$

$M(h)$ is a linear combination of the integrals I_n. These integrals can be expressed in terms of I_0 and I_1 [13]. In particular we get for I_3 [8] and I_4 [13]:

$$I_3=-6hI_0/11+15I_1/11, \tag{2.4.2}$$

$$I_4=21I_0/13-12hI_1/13. \tag{2.4.3}$$

Using (2.4.2) and (2.4.3) we get for $M(h)$:

$$M(h)=(v_1-6v_3h/11+21/13)\,I_0(h)+(v_2+15v_3/11-12h/13)\,I_1(h). \tag{2.4.4}$$

Properties of $I_o(h)$ and $I_1(h)$.

(i) $I_0(h)>0$ for $-2/3<h\leqslant 2/3$ and $I_0(-2/3)=I_1(-2/3)=0$.

(ii) $\lim_{h\to -2/3} I_1(h)/I_0(h)=-1$. (2.4.5)

This follows from the fact that $\int_{\gamma_h} xy\,dx=x^*\int_{\gamma_h} y\,dx$ for some $x^*\in[-2,1]$. As $h\to -2/3\, x^*\to -1$.

Hence, instead of studying $M(h)$, we can study

$$\bar{M}(h)=M(h)/I_0;$$

$$\bar{M}(h)=(v_1-6hv_3/11+21/13)-(v_2+15v_3/11-12h/13)\,P(h)$$
$$=A(h)-B(h)\,P(h), \tag{2.4.6}$$

where

$$P(h)=-I_1(h)/I_0(h) \qquad \text{for } -2/3<h<2/3,\ P(-2/3)=1, \tag{2.4.7}$$

$$A(h)=v_1-6hv_3/11+21/13, \tag{2.4.8}$$

$$B(h)=v_2+15v_3/11-12h/13. \tag{2.4.9}$$

For $-2/3<h<2/3$ we have $M(h)=M'(h)=\cdots=M^{(k)}(h)=0$, $M^{(k+1)}(h)\neq 0$ if and only if $\bar{M}(h)=\bar{M}'(h)=\cdots=\bar{M}^{(k)}(h)=0$, $\bar{M}^{(k+1)}(h)\neq 0$.

We recall the following properties of the function P [8]:

(i) $P([-2/3, 2/3])\subset[5/7, 1]$, $P(-2/3)=1$, $P(2/3)=5/7$; (2.4.10)

(ii) $P'(h) < 0$ for $h \in [-2/3, 2/3)$, $P'(-2/3) = -1/8$,
$P'(2/3) = -\infty$; (2.4.11)

(iii) $P''(h) < 0$ for $h \in [-2/3, 2/3)$, $P''(-2/3) = -55/1152$. (2.4.12)

There is an error of factor 2 in $P''(-2/3)$ in [8]. These properties are deduced from the fact that P satisfies the following differential equation:

$$\text{(iv)} \quad (9h^2 - 4) P' = 7P^2 + 3hP - 5. \tag{2.4.13}$$

This is a Ricatti equation.

(v) We rewrite property (iv) in the following form: (P, h) is a solution of the following system of equations [8]:

$$\begin{aligned} \dot{P} &= -7P^2 - 3hP + 5 \\ \dot{h} &= 4 - 9h^2. \end{aligned} \tag{2.4.14}$$

This system has a saddle point at $(-2/3, 1)$ and an attractive node at $(2/3, 5/7)$ (Fig. 5). The graph of the function $P(h)$ is the unstable separatrix of $(-2/3, 1)$ in system (2.4.14). It joins the point $(-2/3, 1)$ to the point $(2/3, 5/7)$.

From the foregoing we deduce the fundamental theorem:

THEOREM 2.4.1. *System (1.9) has at most three limit cycles for all sufficiently small values of the parameters μ.*

Proof. For any given μ, i.e., ν, we must count the number of zeros in $[-2/3, 2/3]$ of $\bar{M}(h) = A(h) - B(h) P(h)$, with $A(h)$ and $B(h)$ given in (2.4.8) and (2.4.9). Let $h = h^*$ be the zero of $B(h)$.

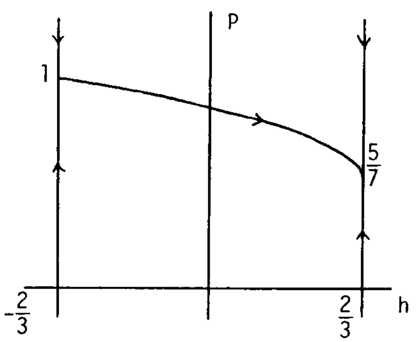

FIG. 5. Phase portrait of (2.4.14). The trajectory joining $(-\frac{2}{3}, 1)$ to $(\frac{2}{3}, \frac{5}{7})$ is the graph of $P(h)$.

(i) Let us suppose first that $h^* \in [-2/3, 2/3]$ is a zero of $\bar{M}(h)=0$. Then we must have $A(h^*)=0$, i.e., $\bar{M}(h)=C(h-h^*)(P-P^*)$. Since P is monotonic, $\bar{M}(h)$ has at most two zeros.

(ii) For the rest of the proof, we can suppose that h^* defined above is not a zero of $\bar{M}(h)=0$. Then,

$$\bar{M}(h)=B(h)(Q(h)-P(h)), \tag{2.4.15}$$

where

$$Q(h)=A(h)/B(h)=(143v_1-78v_3h+231)/(143v_2+195v_3-132h). \tag{2.4.16}$$

We must count the number of intersection points of $P=P(h)$, called Γ_P, and $P=Q(h)$, called Γ_Q, for $h \in [-2/3, 2/3]$. The curve $Q(h)$ is a hyperbola (Fig. 6) and

$$Q'(h)=\alpha/B^2(h), \tag{2.4.17}$$

with

$$\alpha=-78v_3(143v_2+195v_3)+132(143v_1+231). \tag{2.4.18}$$

In the case $\alpha>0$, or in the case $\alpha<0$ and $h^* \leqslant -2/3$ (Fig. 7a), Γ_P and Γ_Q have at most two intersection points.

(iii) In the case $-2/3<h^* \leqslant 2/3$, the right branch Γ_Q^+ of Γ_Q always has at most two intersection points with Γ_P (Figs. 7b and 7c). It has exactly one intersection point if and only if $Q(2/3)<5/7$. In the case where Γ_Q^+ intersects Γ_p, the left branch Γ_Q^- of Γ_Q has no intersection with Γ_p. Therefore we need only consider the case where Γ_Q^+ does not intersect Γ_p. In that case we count the number of intersection points of Γ_Q^- and Γ_p (Figs. 7d and 7e).

For this purpose we count the number of contact points of Γ_Q with the vector field (2.4.14), i.e., we consider the number of zeros of

$$\dot{G}=dG/dt\,|_{(2.4.14)}=\dot{P}-Q'(h)\,\dot{h}, \tag{2.4.19}$$

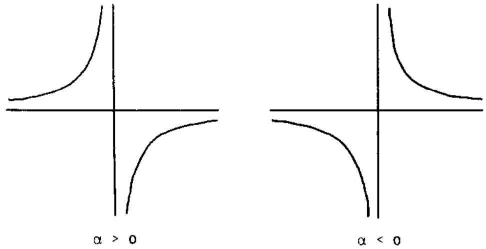

FIG. 6. The curve Γ_Q.

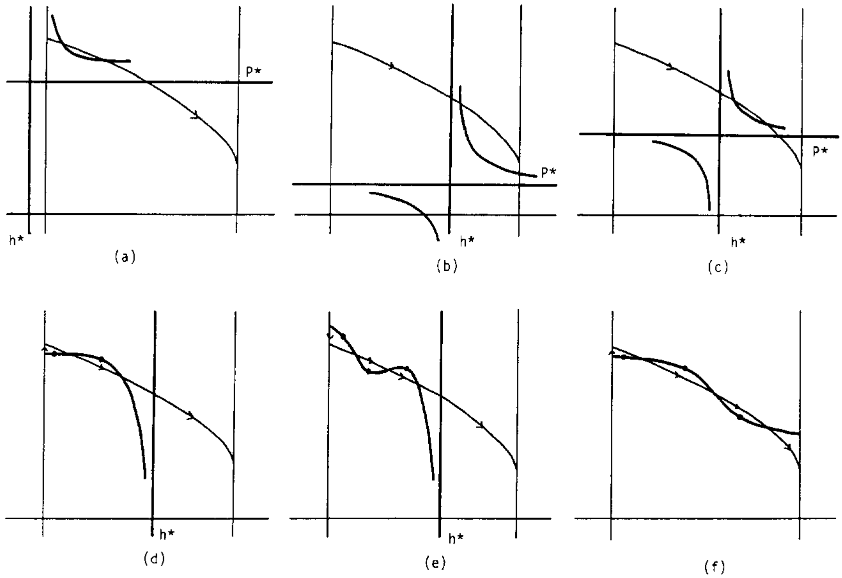

FIG. 7. Relative positions of Γ_P and Γ_Q.

for $G(h) = P(h) - Q(h)$;

$$\begin{aligned}\dot{G} &= (-7P^2 - 3hP + 5) - \alpha(4 - 9h^2)/B^2 |_{P = A/B} \\ &= [-7A^2 - 3hAB + 5B^2 - \alpha(4 - 9h^2)]/B^2 \\ &= (-7Q^2 - 3hQ + 5) - \alpha(4 - 9h^2)/B^2. \end{aligned} \tag{2.4.20}$$

The numerator is a polynomial of degree 3 with respect to h, and thus has at most three roots. It is easily checked hat the left branch Γ_Q^- has at least as many contact points with the vector field (2.4.14) as it has intersection points with Γ_P (Figs. 7d and 7e): between any two intersection points there is always a contact point, and there is always a contact point on the left of the first intersection point, due to the direction of the vector field at the intersection of Γ_Q^- with $h = -2/3$. The number of intersection points is therefore at most three.

(iv) We now consider the case $h^* > 2/3$. By the same argument as in (iii), Γ_Q^- has at most three contact points with the field (2.4.14). Since there is always a contact point to the left of the first intersection point between Γ_Q^- and Γ_P, then Γ_Q^- and Γ_P have at most three intersection points (for example, see Fig. 7f).

2.5. *Determination of the Double and Triple Limit Cycles*

The set of parameter values for which we have a multiple limit cycle is given by the surface (2C) with equation $\bar{M}(h)=0$, $\bar{M}'(h)=0$, for μ sufficiently small. The points of this surface satisfying $\bar{M}''(h)\neq 0$ correspond to double limit cycles. We show that the surface is regular at these points. We also prove that the points of (2C) satisfying $\bar{M}''(h)=0$ form a smooth curve (3C), on which we have $\bar{M}'''(h)\neq 0$. $\bar{M}'(h)$, $\bar{M}''(h)$, and $\bar{M}'''(h)$ are given by

$$\bar{M}'(h)= -6v_3/11+12P(h)/13-B(h)\,P'(h), \tag{2.5.1}$$

$$\bar{M}''(h)=24P'(h)/3-B(h)\,P''(h), \tag{2.5.2}$$

$$\bar{M}'''(h)=36P''(h)/13-B(h)\,P'''(h). \tag{2.5.3}$$

PROPOSITION 2.5.1. *(2C) is a regular surface at points where* $\bar{M}''(h)\neq 0$.

Proof. From $\bar{M}'(h)=0$, $\bar{M}''(h)\neq 0$ we have $h=h(v_2, v_3)$ by the implicit function theorem. Since $\partial\bar{M}/\partial v_1\neq 0$, from $\bar{M}(h)=0$ we have $v_1=v_1(h, v_2, v_3)$. Replacing h by $h(v_2, v_3)$, we arrive at the proposition. ∎

PROPOSITION 2.5.2. *On the curve (3C) corresponding to points where* $\bar{M}(h)=\bar{M}'(h)=\bar{M}''(h)=0$, *we have* $\bar{M}'''(h)\neq 0$.

Proof. The first step is to prove that $\bar{M}'''(h)\neq 0$ if and only if

$$3[P''(h)]^2-2P'(h)\,P'''(h)\neq 0. \tag{2.5.4}$$

This quantity will be shown to be negative. From (2.5.2) we get that $\bar{M}''(h)=0$ is equivalent to

$$B(h)=24P'/(13P''). \tag{2.5.5}$$

Substituting (2.5.5) in (2.5.3) we get

$$\bar{M}'''(h)=12[3P''^2-2P'P''']/13. \tag{2.5.6}$$

We now show $\bar{M}'''(h)\neq 0$. From (2.4.13) we get

$$\begin{aligned}(9h^2-4)^2P'' &= (14P-15h)(7P^2+3hP-5)+3P(9h^2-4)\\ &= 98P^3-63hP^2-18h^2p-82P+75h, \qquad (2.5.7)\\ (9h^2-4)^3P''' &= (14P'-12)(9h^2-4)(7P^2+3hP-5)\\ &\quad +(14P-33h)(9h^2-4)^2P''. \qquad (2.5.8)\end{aligned}$$

This implies that

$$(9h^2-4)^3(3P''^2-2P'P''')$$
$$=637P^4-462hP^3-72h^2P^2+690hP-178P^2-275, \quad (2.5.9)$$

after a few calculations. We call $G(h, P)$ the function:

$$G(h, P)=637P^4-462hP^3-72h^2P^2+690hP-178P^2-275. \quad (2.5.10)$$

We are interested to show that $G(h, P) \neq 0$ on the graph of $P(h)$, except for $h = \pm 2/3$. (Note that P is used here in two ways, as an independent variable, and as a function $P(h)$ satisfying Eq. (2.4.13).)

We first prove that $G(h, P)=0$ determines a unique branch curve $\bar{p}(h)$ in the rectangle $[-2/3, 2/3] \times [5/7, 1]$, passing through the points $(-2/3, 1)$ and $(2/3, 5/7)$. For this we first show that $G(h, P)=G'_P(h, P)=0$ has no solution for $-2/3 \leqslant h \leqslant 2/3$ and $5/7<P<1$. Since

$$G'_P(h, P)=2548P^3-1386hP^2-144h^2P-356P+690h, \quad (2.5.11)$$

$$PG'_P(h, P)-4G(h, P)=462P^3h+144P^2h^2+356P^2-2070Ph+1100, \quad (2.5.12)$$

(2.5.12) is strictly positive for $h \leqslant 0$ and $5/7<P<1$. In the case $h>0$ we consider instead the quantity:

$$PG'_P(h, P)-G(h, P)=P^3(1911P-924h)+(275-72P^2h^2-178P^2). \quad (2.5.13)$$

Both quantities inside parentheses are strictly positive for $5/7<P<1$ and $0<h \leqslant 2/3$.

We have found that along $G(h, P)=0$, $G'_P>0$. So $G(h, P)=0$ locally defines a curve $P=\bar{p}(h)$. We have also

$$G'_h=P(-462P^2-144Ph+690)>0, \quad (2.5.14)$$

$$\bar{p}'(h)=-G'_h/G'_P<0. \quad (2.5.15)$$

To be sure that $G(h, P)=0$ has a unique branch in $[-2/3, 2/3] \times [5/7, 1]$, joining $(-2/3, 1)$ to $(2/3, 5/7)$ we need only check that on $P=1$ (resp. $P=5/7$) the only point of $G(h, P)=0$ is $h=-2/3$ (resp. $h=2/3$). Calculation shows that

$$\bar{p}'(-2/3)=P'(-2/3)=-1/8, \qquad \bar{p}''(-2/3)=P''(-2/3)=-55/1152, \quad (2.5.16)$$

$$\bar{p}'''(-2/3)=-4015/55296<P'''(-2/3)=-3685/73728,$$
$$\bar{p}'(2/3)=-25/56. \quad (2.5.17)$$

So $P(h)$ and $\bar{p}(h)$ have contact of order 3 (resp. 1) at $(-2/3, 1)$ (resp. $(2/3, 5/7)$), and $\bar{p}(h)$ is below $P(h)$ near $h=-2/3$ and $h=2/3$.

We prove that $\bar{p}(h)$ is a curve without contact with respect to the vector field (2.4.14). If this is the case the curve $\bar{p}(h)$ can never lie above the curve $P(h)$. Otherwise, we get a contact point on the curve (Fig. 8). So we suppose

$$\dot{G} = dG/dt\,|_{(2.4.14)} = G'_P \dot{P} + G'_h \dot{h} = 0 \qquad \text{at a point } P = \bar{p}(h);$$

$$\dot{G} = 17836P^5 - 2058P^4h - 9324P^3h^2 - 13384P^3 + 11268P^2h - 1728h^3P^2 + 9000h^2P - 980P - 3450h = 0; \tag{2.5.18}$$

$$(364P - 222h)\,G - 13\dot{G} = 60F(h, P), \tag{2.5.19}$$

with

$$F(h, P) = 126P^3h^2 - 1820P^3 - 1086P^2h - 108h^3P^2 - 603Ph^2 + 1456P + 270h. \tag{2.5.20}$$

We take polar coordinates $h = r\cos\theta$, $P = r\sin\theta$. In the rectangle $-2/3 < h < 2/3$, $5/7 < P < 1$, we have $-2/3 < \operatorname{ctg}\theta < 14/15$;

$$G = r^4\sin^2\theta(637\sin^2\theta - 462\sin\theta\cos\theta - 72\cos^2\theta) - r^2\sin\theta(178\sin\theta - 690\cos\theta) - 275 = 0, \tag{2.5.21}$$

$$F/r = r^4\sin^2\theta\cos^2\theta(126\sin\theta - 108\cos\theta) - r^2\sin\theta(1820\sin^2\theta + 1086\sin\theta\cos\theta + 603\cos^2\theta) + (1456\sin\theta + 270\cos\theta) = 0; \tag{2.5.22}$$

$$\cos^2\theta(126\sin\theta - 108\cos\theta)\,G - (637\sin^2\theta - 462\sin\theta\cos\theta - 72\cos^2\theta)\,F/r = 91[r^2\sin\theta\,\varphi(\cos\theta, \sin\theta) - \psi(\cos\theta, \sin\theta)] = 0, \tag{2.5.23}$$

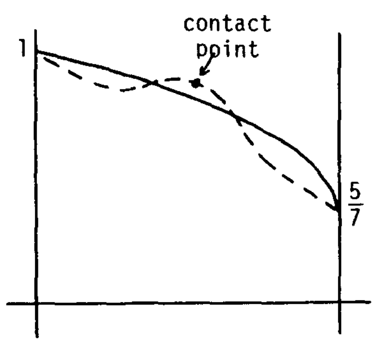

FIG. 8. If $\bar{p}(h)$ lies above $P(h)$ at some point, then $\bar{p}(h)$ has a contact point with (2.4.14).

where

$$\varphi(\cos\theta, \sin\theta) = 12{,}740\sin^4\theta - 1638\sin^3\theta\cos\theta - 2979\sin^2\theta\cos^2\theta$$
$$- 2754\sin\theta\cos^3\theta - 1296\cos^4\theta, \tag{2.5.24}$$

$$\psi(\cos\theta, \sin\theta) = 10{,}192\sin^3\theta - 5502\sin^2\theta\cos\theta$$
$$- 2142\sin\theta\cos^2\theta - 540\cos^3\theta. \tag{2.5.25}$$

Noting that $\sin\theta > 0$ and letting $\xi = \operatorname{ctg}\theta \in (-2/3, 14/15)$ we remark that

$$\varphi(\cos\theta, \sin\theta) = \sin^4\theta(12{,}740 - 1638\xi - 2979\xi^2 - 2754\xi^3 - 1296\xi^4) > 0, \tag{2.5.26}$$

$$\psi(\cos\theta, \sin\theta) = \sin^3\theta(10{,}192 - 5502\xi - 2142\xi^2 - 540\xi^3) > 0. \tag{2.5.27}$$

We obtain $r^2 = \psi(\cos\theta, \sin\theta)/[\sin\theta\,\varphi(\cos\theta, \sin\theta)]$ from (2.5.23), substitute it into (2.5.21), and get

$$(637 - 462\xi - 72\xi^2)\,\psi^2(\xi, 1)$$
$$- (178 - 690\xi)\,\psi(\xi, 1)\,\varphi(\xi, 1) - 275\varphi^2(\xi, 1) = 0. \tag{2.5.28}$$

This gives the identity:

$$Q(\xi) = 552{,}698{,}640\xi^7 + 3{,}491{,}213{,}076\xi^6 - 6{,}227{,}071{,}344\xi^5$$
$$- 5{,}996{,}012{,}607\xi^4 + 7{,}995{,}706{,}992\xi^3 + 4{,}714{,}314{,}696\xi^2$$
$$- 2{,}912{,}547{,}456\xi - 1{,}577{,}629{,}872 = 0. \tag{2.5.29}$$

$Q(\xi)$ can be factorized (using Macsyma) as

$$Q(\xi) = 3159(3\xi + 2)^3(6\xi - 7)^2(12\xi + 91)(15\xi - 14), \tag{2.5.30}$$

which has no real root for $-2/3 < \xi < 14/15$. ∎

THEOREM 2.5.3. *For sufficiently small μ the curve $(3C)$ is a smooth curve corresponding to a triple limit cycle.*

Proof. The fact that (3C) corresponds to a triple limit cycle follows from the definition of (3C) as the set of μ such that $\bar{M}(h) = \bar{M}'(h) = \bar{M}''(h) = 0$, and Proposition 2.5.2 which ensures us that $\bar{M}'''(h) \neq 0$.

From (2.4.6), (2.5.1), and (2.5.2), $\bar{M}(h) = \bar{M}'(h) = \bar{M}''(h) = 0$ is equivalent to

$$\begin{aligned} v_1 &= 3(4hP - 8hP'^2/P'' + 8P'P/P'' - 7)/13, \\ v_2 &= 3(4h - 10P + 20P'^2/P'' + 8P'/P'')/13, \\ v_3 &= 22(P - 2P'^2/P'')/13, \end{aligned} \tag{2.5.31}$$

and Eq. (2.5.4) yields $\partial v_3/\partial h \neq 0$. From this we get $h = h(v_3)$. The two first equations in (2.5.31) give v_1 and v_2 as functions of h. Replacing h by $h(v_3)$ gives the desired result. ▮

2.6. *Relative Positions of Hopf, Homoclinic, and Multiple Limit Cycle Bifurcations*

THEOREM 2.6.1. *For μ sufficiently small, there is a smooth curve $(H, 2C)$ in the parameter space corresponding to the simultaneous occurrence of a Hopf bifurcation of order 1 and a double limit cycle. This curve joins $(v_1, v_2, v_3) = (11/5, 2/5, 14/5)$, corresponding to Hopf bifurcation of order 3(H_3) to the point $(v_1, v_2, v_3) = (-1, 22/13, -22/13)$, corresponding to the simultaneous occurrence of Hopf bifurcation of order 1 and homoclinic loop bifurcation of order 2 (H, HL_2). The curve is the convex envelope of the family of lines $\bar{M}(h) = 0$, $h \in [-2/3, 2/3]$, in the Hopf bifurcation plane.*

Proof. We separate the proof into two parts. In the first part we explain the ideas underlying the proof. The second part consists of calculations, in the same spirit as Proposition 2.5.2, and can be skipped on a first reading.

(i) We make a change of coordinates, which brings the two lines (H_2) and (H, HL) to coordinate axes:

$$\begin{aligned} m_1 &= v_1 - v_2 - v_3 + 1 \\ m_2 &= v_2 - 3v_3 + 8 \\ m_3 &= v_1 - 5v_2/7 - 103v_3/77 + 187/91. \end{aligned} \tag{2.6.1}$$

The equation $\bar{M}(h) = 0$ (Eq. (2.4.6)), under the condition $m_1 = 0$, gives

$$\begin{aligned} \bar{M}(h) = {} & m_2(14P + 3h - 12)/10 + 7m_3(22 - 24P - 3h)/20 \\ & + 4(34 - 38P - 21h + 15hP)/65 = 0. \end{aligned} \tag{2.6.2}$$

So

$$m_2 = 7m_3Q(h)/2 + 8R(h)/13, \tag{2.6.3}$$

with

$$Q(h) = (3h + 24P - 22)/(3h + 14P - 12), \tag{2.6.4}$$

$$R(h) = (-34 + 38P + 21h - 15hP)/(3h + 14P - 12). \tag{2.6.5}$$

Since $Q < 0$ and $R > 0$, we show that $dQ/dh \neq 0$, $d^2R/dQ^2 < 0$, and hence that $(-R)$ is a convex function of the slope Q. Therefore the curve (H, 2C) is the graph of the Legendre transform of $(-R)$, i.e., a convex curve.

(i) We have

$$dQ/dh = 10[3(1-P) + P'(3h+2)]/(3h + 14P - 12)^2 < 0, \tag{2.6.6}$$

except at $h = -2/3$, since the numerator is zero at $h = -2/3$ and its derivative $P''(3h+2) < 0$, and

$$dR/dh = 5[-30 + 72P - 42P^2 + (4 - 9h^2)\, P']/(3h + 14P - 12)^2. \tag{2.6.7}$$

So

$$2dR/dQ = [-30 + 72P - 42P^2 + (4 - 9h^2)\, P']/[3(1-P) + P'(3h+2)], \tag{2.6.8}$$

$$2d^2R/dh\, dQ = 3(12 - 3h - 14P)[P''(1-P)(3h+2) + 6P'(1-P) + 2P'^2(3h+2)]/[3(1-P) + P'(3h+2)]^2. \tag{2.6.9}$$

Since $(12 - 3h - 14P)$ is negative everywhere, except at $h = \pm 2/3$, we must show that the quantity $G = P''(1-P)(3h+2) + 6P'(1-P) + 2P'^2(3h+2) < 0$ on $(-2/3, 2/3)$. Since $G(-2/3) = G'(-2/3) = G''(-2/3) = 0$, and $G'''(-2/3) = 3(3P''^2 - 2P'P''')|_{h=-2/3} < 0$, it is enough to show that $G \neq 0$ on $(-2/3, 2/3)$. We study instead

$$A = (3h-2)(9h^2-4)[P''(1-P)(3h+2) + 6P'(1-P) + 2P'^2(3h+2)]. \tag{2.6.10}$$

Using (2.4.13) and (2.5.7), we get

$$A = (14P + 3h - 12)(7P^2 + 3hP - 5)(1-P) + 3P(1-P)(9h^2-4) + 2(7P^2 + 3hP - 5)^2, \tag{2.6.11}$$

and we study the curve defined by $A(h, P) = 0$. Since

$$dA/dP = (14P + 3h)(3hP + 3h + 26P - 22) + (7P^2 + 3hP - 5)(3h + 26) + 3(1 - 2P)(9h^2 - 4) > 0, \tag{2.6.12}$$

for $-2/3<h<2/3$, and $5/7<P<1$, $A(h, P)=0$ defines $P=a(h)$ locally. $A|_{P=1}=0$ has a double root at $h=-2/3$; $A|_{P=5/7}$ has roots $h=2/3$ and $h=2/9$; $A|_{h=-2/3}$ has a double root at $P=1$ and a simple root at $P=-5/7$; $A|_{h=2/3}$ has a double root at $P=5/7$ and a simple root at $P=-1$. So $A(h, P)=0$ defines three branches $P=a_i(h), i=1, 2, 3$, $-2/3<h<2/3$ (see Fig. 9). We can check that

$$\begin{aligned}
&a_1'(-2/3)=-1/8=P'(-2/3),\\
&a_1''(-2/3)=-55/1152=P''(-2/3),\\
&a_1'''(-2/3)=-55/756<P'''(-2/3)=-3685/73{,}728. \qquad (2.6.13)
\end{aligned}$$

So $P=a_1(h)$ lies below $P=P(h)$ near $h=-2/3$. In order to check the situation near $h=2/3$, we change $P=P(h)$ and $P=a_1(h)$ to $h=h(P)$ and $h=\bar{h}(P)$, respectively. Calculation shows that

$$\begin{aligned}
&h'(5/7)=0=\bar{h}'(5/7),\\
&h''(5/7)=0, \qquad \bar{h}''(5/7)=-2744/699. \qquad (2.6.14)
\end{aligned}$$

Hence $P=a_1(h)$ lies below $P=P(h)$ near $h=+2/3$.

As in Proposition 2.5.2, in order to prove that $P=a_1(h)$ and $P=P(h)$

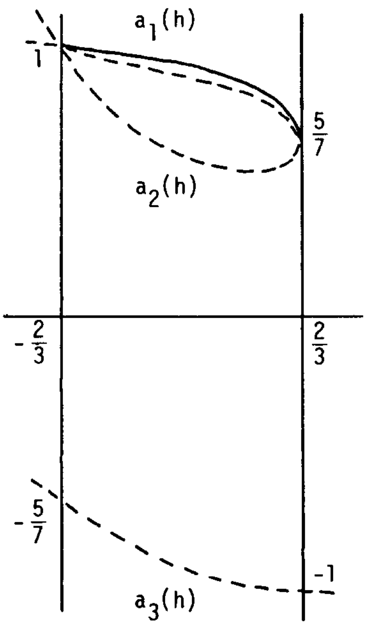

FIG. 9. The curves $P=a_i(h)$ given by $A=0$ in (2.6.11).

have no intersection points, we now prove that $P = a_1(h)$ is without contact with the vector field (2.4.14); i.e., we consider the equation $A = \dot{A} = 0$,

$$\begin{aligned} -\dot{A}|_{(2.4.14)} = {} & 3(7P^2 + 3hP - 5)(1 + 3P)(9h^2 - 4) \\ & + 3P(1 - P)(14P + 21h - 12)(9h^2 - 4) \\ & + (7P^2 + 3hP - 5)^2(26 + 28P + 9h) \\ & + 3(1 - 2P)(9h^2 - 4)(7P^2 + 3hP - 5) \\ & + (1 - P)(14P + 3h)(14P + 3h - 12)(7P^2 + 3hP - 5). \quad (2.6.15) \end{aligned}$$

$A = 0$ gives

$$\begin{aligned} & 3P(1 - P)(9h^2 - 4) \\ & \quad = -(7P^2 + 3hP - 5)[(14P + 3h - 12)(1 - P) + 2(7P^2 + 3hP - 5)]. \end{aligned} \quad (2.6.16)$$

So

$$\begin{aligned} -\dot{A} = {} & (7P^2 + 3hP - 5)[3(2 + P)(9h^2 - 4) + 6(14P + 3h - 12)(1 - P)(2 - 3h) \\ & + (7P^2 + 3hP - 5)(50 - 33h)]. \end{aligned} \quad (2.6.17)$$

We let

$$\begin{aligned} B = {} & 3(2 + P)(9h^2 - 4) + 6(14P + 3h - 12)(1 - P)(2 - 3h) \\ & + (7P^2 + 3hP - 5)(50 - 33h). \end{aligned} \quad (2.6.18)$$

We need only show that $A = B = 0$ has no solution for $-2/3 < h < 2/3$, and $5/7 < P < 1$. We expand A and B:

$$\begin{aligned} A = {} & 21P^3h - 18P^2h^2 + 182P^3 + 99P^2h \\ & + 36Ph^2 - 142P^2 - 81Ph - 142P - 15h + 110, \end{aligned} \quad (2.6.19)$$

$$B = 21P^2h - 18Ph^2 + 182P^2 - 354hP + 300P + 417h - 418. \quad (2.6.20)$$

Let

$$C = A - PB = 453P^2h + 36Ph^2 - 442P^2 - 498Ph + 276P - 15h + 110, \quad (2.6.21)$$

$$\begin{aligned} D = (55B + 209C)/66 = {} & 1452P^2h + 99Ph^2 \\ & - 1248P^2 - 1872Ph + 1124P + 300h. \end{aligned} \quad (2.6.22)$$

In polar coordinates $h = r\cos\theta$, $P = r\sin\theta$, B, C, and D become

$$B = 3\cos\theta\sin\theta(7\sin\theta - 6\cos\theta)\,r^3 + \sin\theta(182\sin\theta - 354\cos\theta)\,r^2 \\ + (300\sin\theta + 417\cos\theta)\,r - 418, \tag{2.6.23}$$

$$C = 3\cos\theta\sin\theta(151\sin\theta + 12\cos\theta)\,r^3 - \sin\theta(442\sin\theta + 498\cos\theta)\,r^2 \\ + (276\sin\theta - 15\cos\theta)\,r + 110, \tag{2.6.24}$$

$$D = 33\cos\theta\sin\theta(44\sin\theta + 3\cos\theta)\,r^2 - 624\sin\theta(2\sin\theta + 3\cos\theta)\,r \\ + 4(281\sin\theta + 75\cos\theta). \tag{2.6.25}$$

Let

$$\begin{aligned} E &= (151\sin\theta + 12\cos\theta)\,B - (7\sin\theta - 6\cos\theta)\,C \\ &= 12\sin\theta(2548\sin^2\theta - 4203\sin\theta\cos\theta - 603\cos^2\theta)\,r^2 \\ &\quad + (43{,}368\sin^2\theta + 68{,}328\sin\theta\cos\theta + 4914\cos^2\theta)\,r \\ &\quad + (-63{,}888\sin\theta - 4356\cos\theta). \end{aligned} \tag{2.6.26}$$

We consider now the equations $D = E = 0$, instead of $A = B = 0$. Let

$$\begin{aligned} F &= -4(2548\sin^2\theta - 4203\sin\theta\cos\theta - 603\cos^2\theta)\,D \\ &\quad + 11(44\sin\theta + 3\cos\theta)\,E \\ &= 182(f_1(\theta)\,r - f_2(\theta)), \end{aligned} \tag{2.6.27}$$

where

$$f_1(\theta) = 3(23{,}296\sin^4\theta + 34{,}960\sin^3\theta\cos\theta + 36\sin^2\theta\cos^2\theta \\ + 216\sin\theta\cos^3\theta + 297\cos^4\theta), \tag{2.6.28}$$

$$f_2(\theta) = 2(31{,}472\sin^3\theta + 41{,}436\sin^2\theta\cos\theta \\ - 9720\sin\theta\cos^2\theta - 1593\cos^3\theta). \tag{2.6.29}$$

Substituting $r = f_2(\theta)/f_1(\theta)$ into $D = 0$, multiplying by $f_1^2(\theta)$, and letting $\xi = \operatorname{ctg}\theta$, we obtain finally:

$$7260\xi(3\xi + 2)^5(3\xi + 44)^2(15\xi - 14) = 0 \tag{2.6.30}$$

(the factorization was obtained on Macsyma). This expression has only one root: $\xi = 0$, for $-2/3 < \xi < 14/15$. This root corresponds to $h = 0$. But it is easily verified that $A|_{h=0} = B|_{h=0} = 0$ (or equivalently $B|_{h=0} = C|_{h=0} = 0$) has no solution for $5/7 < P < 1$. ∎

THEOREM 2.6.2. *For sufficiently small μ, there is a smooth curve $(HL, 2C)$ in the parameter space, corresponding to the simultaneous occurrence of a homoclinic loop bifurcation of order* 1 *and a double limit cycle. This curve joins the point* $(\nu_1, \nu_2, \nu_3) = (-107/91, -94/91, 110/91)$, *corresponding to a homoclinic loop bifurcation of order* 3 (HL_3), *to the point* $(\nu_1, \nu_2, \nu_3) = (31/65, -58/65, 154/65)$ *corresponding to the simultaneous occurrence of a homoclinic bifurcation of order* 1 *and a Hopf bifurcation of order* $2(H_2, HL)$. *The curve is the convex envelope of the family of lines in the homoclinic loop bifurcation plane, given by* $\bar{M}(h) = 0$, $h \in [-2/3, 2/3]$.

Proof. (i) Idea underlying the proof: We make the change of coordinates

$$\begin{aligned} m_1 &= -\nu_1 + \nu_2 + \nu_3 - 1 \\ m_2 &= \nu_1 - 5\nu_2/7 - 103\nu_3/77 + 187/91 \\ m_3 &= \nu_1 + \nu_2 + \nu_3 + 1. \end{aligned} \tag{2.6.31}$$

The equation $\bar{M}(h) = 0$ (Eq. (2.4.6)), under the condition $m_2 = 0$, gives

$$\bar{M}(h) = m_1(3h-2)/4 + m_3(12 - 3h - 14P)/24 + 4(2-3h)(1-P)/13 = 0. \tag{2.6.32}$$

So

$$m_1 = m_3(14 + 3h - 12)/6(3h-2) + 16(1-P)/13. \tag{2.6.33}$$

We let

$$Q = (14P + 3h - 12)/(3h - 2) \qquad \text{and} \qquad R = 1 - P. \tag{2.6.34}$$

Since $Q < 0$ and $R > 0$, we show $dQ/dh \neq 0$, $d^2R/dQ^2 < 0$, and hence that $(-R)$ is a convex function of the slope Q. Therefore the curve (H, 2C) is the graph of the Legendre transform of $(-R)$, i.e., a convex curve.

(ii) Calculation of dQ/dh, d^2R/dQ^2:

$$dQ/dh = [14P'(3h-2) - 3(14P - 10)]/(3h-2)^2. \tag{2.6.35}$$

The numerator is zero for $h = 2/3$, using the asymptotic expansion of P' (Lemma 2.6.4). The derivative of the numerator is $14P''(3h-2) > 0$. So $dQ/dh < 0$ except at $h = 2/3$, and

$$dR/dQ = P'(3h-2)^2/[14P'(2-3h) + 3(14P - 10)] < 0, \tag{2.6.36}$$

$$d^2R/dh\,dQ = \frac{3(3h-2)[P''(3h-2)(7P-5) + 14P'^2(2-3h) + 6P'(7P-5)]}{2[7P'(2-3h) + 3(7P-5)]^2}. \tag{2.6.37}$$

So we must show $A<0$, with

$$A=P''(3h-2)(7P-5)+14P'^2(2-3h)+6P'(7P-5). \quad (2.6.38)$$

We have already $A(-2/3)<0$. Accordingly it is enough to show that A does not change sign. We consider instead

$$B=(9h^2-4)(3h+2)\,A. \quad (2.6.39)$$

Using (2.4.13) and (2.5.7) we get

$$\begin{aligned} B &= (14P-15h)(7P^2+3hP-5)(7P-5)+3P(7P-5)(9h^2-4) \\ &\quad -14(7P^2+3hP-5)^2+6(3h+2)(7P-5)(7P^2+3hP-5) \\ &= (2-3h)[(7P^2+3hP-5)(7P+5)-3P(7P-5)(3h+2)]. \end{aligned} \quad (2.6.40)$$

So we must show that C has constant sign on $P(h)$, where

$$\begin{aligned} C &= (7P^2+3hP-5)(7P+5)-3P(7P-5)(3h+2) \\ &= 49P^3-42hP^2-7P^2+60hP-5P-25; \end{aligned} \quad (2.6.41)$$

$$dC/dP=147P^2-84hP-14P+60h-5>0, \quad (2.6.42)$$

$$dC/dh=\ -6P(7P-10)>0. \quad (2.6.43)$$

By looking to the points of $C=0$ on the boundary of the rectangle $-2/3\leqslant h\leqslant 2/3$, $5/7\leqslant P\leqslant 1$, we see that $C=0$ gives a unique curve $P=\tilde{P}(h)$, with $d\tilde{P}/dh<0$. Moreover

$$\tilde{P}'(-2/3)=\ -1/8, \qquad \tilde{P}''(-2/3)=\ -5/56<\ -55/1152, \qquad \tilde{P}'(2/3)=\ -5/14. \quad (2.6.44)$$

So $\tilde{P}(h)$ is below $P(h)$ near $h=\ -2/3$ and $h=2/3$. We show that $P=\tilde{P}(h)$ is without contact with the field (2.4.14):

$$\begin{aligned} -\dot{C}\,|_{(2.4.14)} &= [(14P+3h)(7P+5)+7(7P^2+3hP-5)-3(7P-5)(3h+2) \\ &\quad -21P(3h+2)](7P^2+3hP-5)-6P(7P-10)(9h^2-4). \end{aligned} \quad (2.6.45)$$

From $C=0$ we get

$$(7P^2+3hP-5)=3P(7P-5)(3h+2)/(7P+5). \quad (2.6.46)$$

So $\dot{C}=0$ is equivalent under (2.4.46) to

$$147P^3-126hP^2-91P^2+150hP-15P-25=0, \quad (2.6.47)$$

after simplifying by $3P(3h+2)(7P+5)$. Subtracting $3C=0$ (C is given in (2.6.41)) from (2.6.47) we get

$$-10(7P^2+3hP-5)=0, \tag{2.6.48}$$

which has the only solutions $h=\pm 2/3$. ∎

Remarks. (1) The Hopf bifurcation equation can be reconstructed from the equations $\bar{M}(-2/3)=0$. The first (resp. second) focal quantity is zero if $\bar{M}'(-2/3)=0$ (resp. $\bar{M}''(-2/3)=0$).

(2) Similarly the homoclinic loop bifurcation equation can be deduced from $\bar{M}(2/3)=0$. The set of parameter values corresponding to homoclinic loop bifurcation of order 2 (resp. 3) can also be reconstructed from the equations $\bar{M}'(h)=0$ (resp. $\bar{M}'(h)=\bar{M}''(h)=0$), in a neighborhood of $h=2/3$. The proof in this case is more complex than in the case of Hopf bifurcation, and thus is given in the following proposition.

PROPOSITION 2.6.3. *The surface* (2C) (*resp. curve* (3C)) *of double* (*resp. triple*) *limit cycles contains the curve* (*resp. point*) *of homoclinic loop bifurcation of order* 2 (*resp.* 3).

Proof. The equation $\bar{M}(2/3)=0$ is equivalent to homoclinic loop bifurcation. Along (2C) we have

$$\bar{M}'(h)= -6v_3/11+12P/13-BP'=0, \tag{2.6.49}$$

with B given in (2.4.9). This is equivalent to

$$\bar{M}'(h)/P'= -6v_3/11P'+12P/13P'-B=0. \tag{2.6.50}$$

The limit when $h\to 2/3$ must be zero. So $B(2/3)=0$, which together with $\bar{M}(2/3)=0$, is equivalent to homoclinic loop bifurcation of order 2. Along (3C) we must also have

$$\bar{M}''(h)=0=24P'/13-BP''. \tag{2.6.51}$$

In (2.6.49) the sum of the two quantities $(-6v_3/11+12P/13)$ and $-BP'$ can be zero without each of the two quantities being zero. We show that, with the additional hypothesis (2.6.51), the two quantities tend to zero when $h\to 2/3$. This is equivalent to homoclinic loop bifurcation of order 3. Suppose that $(-6v_3/11+12P/13)$ and $-BP'$ have a nonzero limit when $h\to 2/3$. Equation (2.6.51) is equivalent to

$$\bar{M}''(h)/P'=24/13-(BP')\,P''/P'^2. \tag{2.6.52}$$

The first term is finite. The first factor of the second term has a non-zero limit. We only need to prove that $P''/P'^2 \to \infty$, when $h \to 2/3$. This is done in the next lemma. ▮

LEMMA 2.6.4. *$P''/P'^k \to \infty$ for any $k \geqslant 1$, when $h \to 2/3$.*

Proof. We use the asymptotic expansion of I_0 and I_1 in the appendix to [8],

$$I_0 = \alpha_0 + \alpha_1 \bar{h} \ln \bar{h} + \alpha_2 \bar{h} + \alpha_3 \bar{h}^2 \ln \bar{h} + \alpha_4 \bar{h}^2 + ..., \tag{2.6.53}$$

$$I_1 = \beta_0 + \beta_1 \bar{h} \ln \bar{h} + \beta_2 \bar{h} + \beta_3 \bar{h}^2 \ln \bar{h} + \beta_4 \bar{h}^2 + ..., \tag{2.6.54}$$

with $\beta_0/\alpha_0 = -5/7$, $\beta_1/\alpha_1 = 1$, and $\bar{h} = 2/3 - h$. Since $P = -I_1/I_0$, we have

$$P' = (I_1 I_0' - I_1' I_0)/I_0^2, \tag{2.6.55}$$

$$P'' = (I_1 I_0'' I_0 - I_1'' I_0^2 + 2 I_0 I_0' I_1' - 2 I_0'^2 I_1)/I_0^3. \tag{2.6.56}$$

The asymptotic behaviour of I_0', I_1', I_0'', I_1'' for $\bar{h} \to 0$, i.e., $h \to 2/3$, is given by

$$I_0' \sim \alpha_1 \ln \bar{h},\ I_1' \sim \beta_1 \ln \bar{h},\ I_0'' \sim \alpha_1/\bar{h},\ I_1'' \sim \beta_1/\bar{h}. \tag{2.6.57}$$

This gives for P' and P'':

$$P' \sim K_1 \ln \bar{h}, \qquad P'' \sim K_2/\bar{h}. \quad ▮ \tag{2.6.58}$$

THEOREM 2.6.5. *The parameter region for which system* (1.9) *has three limit cycles is a "topological* 3-*simplex"* (Fig. 11).

Proof. We consider the 3-simplex $-2/3 < h_1 < h_2 < h_3 < 2/3$, and the mapping F from that 3-simplex to parameter space $v = (v_1, v_2, v_3)$, defined by $\mathbf{h} = (h_1, h_2, h_3) \mapsto v(\mathbf{h})$, where $v(\mathbf{h})$ is the solution of

$$\bar{M}(h_1) = \bar{M}(h_2) = \bar{M}(h_3) = 0. \tag{2.6.59}$$

This solution is unique: the determinant of the system is positive

$$\begin{vmatrix} 1 & -P(h_1) & -6h_1/11\text{-}15/11 \\ 1 & -P(h_2) & -6h_2/11\text{-}15/11 \\ 1 & -P(h_3) & -6h_3/11\text{-}15/11 \end{vmatrix} = 6/11 \begin{vmatrix} 1 & P(h_1) & h_1 \\ 1 & P(h_2) & h_2 \\ 1 & P(h_3) & h_3 \end{vmatrix} \tag{2.6.60}$$

$$= 6/11(h_2 - h_1)(h_3 - h_2)[(P(h_2) - P(h_1))/(h_2 - h_1)$$
$$- (P(h_3) - P(h_2))/(h_3 - h_2)] > 0,$$

since the function P is concave. The function F is a local diffeomorphism on the 3-simplex in $\mathbf{h}$-space. We want to extend F to the closed 3-simplex.

It is obvious how to do so over the faces $h_1 = -2/3$ and $h_3 = 2/3$, and F is invertible on each of these faces. To extend F to the two faces $h_1 = h_2 \neq h_3$ and $h_1 \neq h_2 = h_3$ we define $F(h_1, h_2 = h_1, h_3) = v(\mathbf{h})$ where $v(\mathbf{h})$ is the solution of

$$\bar{M}(h_1) = \bar{M}'(h_1) = \bar{M}(h_3) = 0, \tag{2.6.61}$$

and similarly for the other face. A similar definition can also be given for all edges. F has rank at least 1 on all edges, and the restriction of F to each edge is a local diffeomorphism. Since system (1.9) cannot have more than three limit cycles (Theorem 2.4.1), i.e., no four planes $\bar{M}(h_i) = 0$, $h_1 < h_2 < h_3 < h_4$, can intersect, F is a global diffeomorphism on the 3-simplex. ▌

This suggests the following definition of a topological 3-simplex.

DEFINITION 2.6.6. We call a *topological n-simplex* the image of $\Delta = \{x | 0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1\}$ under the mapping $F: \Delta \to \mathbb{R}^n$, which is a diffeomorphism in the interior of the simplex, such that its restriction to any simplex contained in the boundary of Δ is also a diffeomorphism in the space of appropriate dimension.

We are now interested in describing the structure of the surface (2C) of limit cycles of multiplicity 2 in v-space. To do this we slice the surface with the planes (Π_h) having equations $\bar{M}(h) = 0$ (Fig. 10). Surface (2C) has equation $\bar{M}(h) = \bar{M}'(h) = 0$, and cuts (Π_h) in a line (L_h). Condition $\bar{M}''(h) = 0$ gives a plane transversal to (L_h), so (L_h) has a unique point $C(h)$ lying on the curve (3C) (Fig. 11). Plane (Π_h) has a point on every edge of the "3-simplex region with three limit cycles!" The only way to achieve this in practice is to have (Π_h) tangent to some of the edges: in this

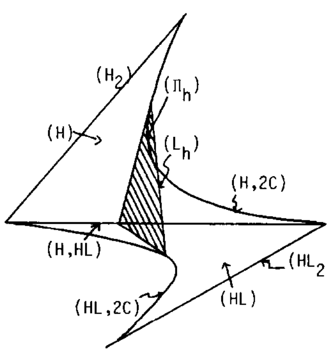

FIG. 10. The triangle of the plane (Π_h) generating the 3-simplex. The line (L_h) generates the surface (2C).

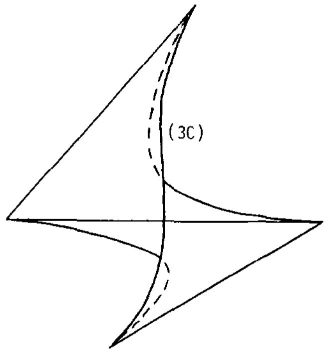

FIG. 11. The 3-simplex region with three limit cycles.

case (Π_h) is tangent to three edges: (H, 2C), (HL, 2C), and (3C). This is possible provided that (3C) is not a planar curve.

An interesting property is that the surface (2C) is a ruled surface, swept by the line (L_h), $h \in [-2/3, 2/3]$. The segment of the line (L_h) between the two points where (L_h) is tangent to (H, 2C) and (HL, 2C) (Fig. 10), generates the two faces of the 3-simplex which lie inside (2C). However the ruled surface (2C) is singular along the curve (3C). The interior of the 3-simplex is swept by the triangle with edges (L_h), $(\Pi_h) \cap (H)$, and $(\Pi_h) \cap (HL)$, with $h \in [-2/3, 2/3]$ (Figs. 10 and 11). This family of triangles can be seen to be parametrized by points of (H, HL). Tangents to the curves (H, 2C) and (HL, 2C) are drawn from one such point, and then the two tangent points are joined. Any point in the simplex belongs to exactly three such triangles.

We have the same kind of geometric interpretation as in the cusp of order 3. For any point in ν-space, the number of limit cycles for that point is equal to the number of planes through that point, which are tangent to both (H, 2C) and (HL, 2C). Since different planes correspond to different values of h, we can see that different points in ν-space cannot have limit cycles of the same size (i.e., corresponding to the same values of h).

3. A REMARK ON THE CUSP OF ORDER 4

In this section we show how the previous results can be used to prove that

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon_1 + \varepsilon_2 y + x^2 + \varepsilon_3 xy + \varepsilon_4 x^3 y \pm x^4 y \end{aligned} \tag{3.1}$$

is a *universal unfolding* of

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 \pm x^4 y. \end{aligned} \tag{3.2}$$

By universal unfolding we mean the following: for every unfolding of (3.2),

$$\begin{aligned} \dot{x} &= F(x, y, \lambda) \\ \dot{y} &= G(x, y, \lambda), \end{aligned} \tag{3.3}$$

with λ a multi-parameter, i.e., $F(x, y, 0) = y$ and $G(x, y, 0) = x^2 \pm x^4 y$, there is a mapping $\varphi(\lambda) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ defined in a neighborhood of zero in λ-space, and a neighborhood V of zero in phase space such that the vector field (3.3) is topologically equivalent to (3.1) for $\varepsilon = \varphi(\lambda)$.

Family (3.1) has two singular points $q_\pm = (\pm\sqrt{-\varepsilon_1}, 0)$ for $\varepsilon_1 < 0$. For $\varepsilon_1 = 0$, we have a saddle-node bifurcation. When $\varepsilon_1 < 0$, we make the change of coordinates:

$$\begin{aligned} &x = \delta^2 u \qquad y = \delta^3 v \qquad \tau = \delta t \\ &\varepsilon_1 = -\delta^4 \qquad \varepsilon_2 = \delta^8 \nu_1 \qquad \varepsilon_3 = \delta^6 \nu_2 \qquad \varepsilon_4 = \delta^2 \nu_3. \end{aligned} \tag{3.4}$$

System (3.1) becomes

$$\begin{aligned} u' &= v \\ v' &= -1 + u^2 + \delta^7 \nu_1 v + \delta^7 \nu_2 u v + \delta^7 \nu_3 u^3 v + \delta^7 u^4 v, \end{aligned} \tag{3.5}$$

where $'$ denotes the derivative with respect to τ. Taking

$$\mu_4 = \delta^7, \qquad \mu_i = \delta^7 \nu_i, \qquad i = 1, 2, 3, \tag{3.6}$$

we find that studying (3.5) is the same as studying (1.9). For each fixed value of δ, we get the bifurcation diagram in ν-space, as discussed in Section 2. This bifurcation diagram is obtained from the one at $\mu_4 = 1$ multiplied by a factor δ^7. It converges to a point as $\delta \to 0$, giving the conic structure.

The bifurcations appearing for $\varepsilon_1 = 0$ must be studied in greater detail. When $\varepsilon_2 \neq 0$ a saddle-node bifurcation occurs. When $\varepsilon_2 = 0$ and $\varepsilon_3 \neq 0$, we have a Bogdanov–Takens bifurcation (BT) [9], with bifurcation diagram given in Fig. 12 for the case $\varepsilon_3 > 0$ (the case $\varepsilon_3 < 0$ is obtained by $y \mapsto -y$, $t \mapsto -t$, $\varepsilon_2 \mapsto -\varepsilon_2$). The two types of Bogdanov–Takens bifurcation (BT) ($\varepsilon_3 > 0$ and $\varepsilon_3 < 0$) are separated by cusps of order 3. The Hopf bifurcation (H) and homoclinic loop bifurcation (HL) surfaces must branch from

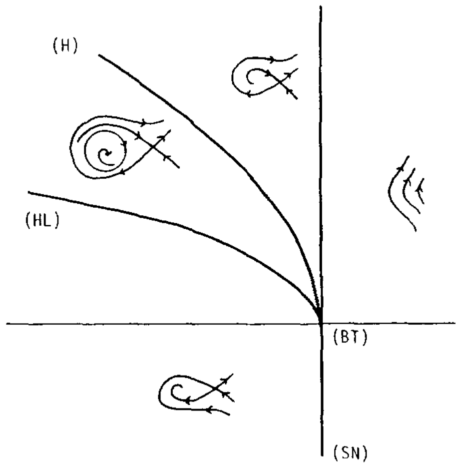

FIG. 12. Bifurcation diagram for the Bogdanov system (1.2). The case with a $-xy$ term can be obtained by $y \mapsto -y$, $t \mapsto -t$, $\varepsilon_2 \mapsto -\varepsilon_2$.

(BT). Half-surfaces (H) and (HL) corresponding to the emergence of an attractive (repulsive) limit cycle, branch from (BT) with $\varepsilon_3 < 0$ (resp. $\varepsilon_3 > 0$).

Finally the case $\varepsilon_2 = \varepsilon_3 = 0$, $\varepsilon_4 \neq 0$, corresponds to cusps of order 3. Their bifurcation diagram, studied in [8], appear in planes transversal to this line.

The bifurcation diagram is described by its intersection with a 3-sphere around the origin in ε-space. The system has singular points only on the closed half 3-sphere $\varepsilon_1 \leqslant 0$, which can be transformed into a closed 3-ball (Fig. 13). The bifurcation diagram inside the ball is the bifurcation diagram of (1.9) in the ν-variables, containing a topological 3-simplex with exactly three limit cycles. The boundary of the ball (i.e., a 2-sphere) corresponds to a saddle-node bifurcation (SN), while Bogdanov–Takens bifurcation appears on a circle on the 2-sphere. Two points of the circle correspond to cusps of order 3 (one with $\varepsilon_4 > 0$, the other with $\varepsilon_4 < 0$), and separate the two types of (BT) ($\varepsilon_3 < 0$ and $\varepsilon_3 > 0$). The surfaces (H) and (HL) inside the ball branch along (BT). Moreover, the codimension 2 bifurcation curves inside the ball meet on the boundary of the ball at the two cusps of order 3, giving the conic structure described in [8]. Figure 13 illustrates the continuity of the codimension 2 curves to the boundary of the ball (for clarity (H) and (HL) are shown only partly).

The proof of the versality involves much work, but essentially all the steps have been done in [8, 13], and we will just recall briefly the main ideas.

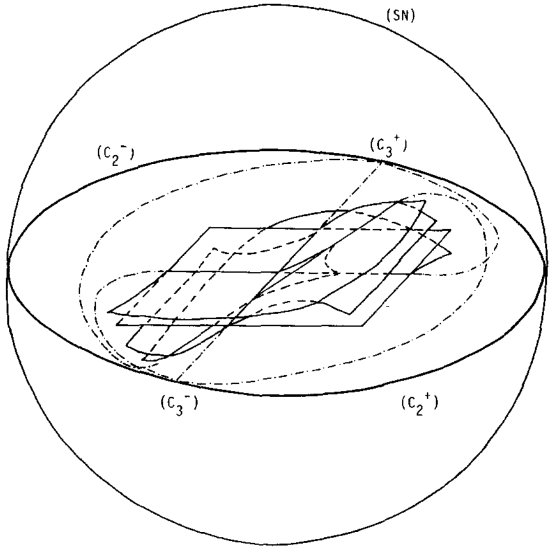

FIG. 13. Intersection of the bifurcation diagram of (3.1) with a 3-sphere around the origin. The closed half 3-sphere, where (3.1) has singular points, is identified to a closed 3-ball. The boundary corresponds to saddle-node bifurcation, and the equator to Bogdanov–Takens bifurcation. Two points of the equator correspond to cusps of order 3. The Hopf bifurcation (H) and homoclinic loop bifurcation (HL) surfaces meet on the equator, corresponding to cusps of order $2(C_2^-)$ and (C_2^+). All codimension 2 bifurcation curves meet at cusps of order 3 (C_3^-) and (C_3^+).

The following steps reduce a general unfolding (3.3) to a family of the form (3.1), plus higher order terms:

(i) Using the implicit function theorem factorize:

$$F(x, y) = (y - f(x, \lambda))\, H(x, y, \lambda) \qquad H(0, 0, 0) \neq 0. \tag{3.7}$$

Make the change of variables $y \mapsto y - f(x, \lambda)$, and divide the system by $H(x, y, \lambda)$. This produces a system,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= G(x, y, \lambda), \end{aligned} \tag{3.8}$$

for which we still call the new coordinates x and y.

(ii) Bring system (3.8) to normal form ([9] for example)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \sum_{i=0}^{5} a_i(\lambda)\, x^i + \sum_{i=0}^{4} b_i(\lambda)\, x^i y + O(|x, y|^6), \end{aligned} \tag{3.9}$$

with $a_2(0) = 1$, $a_i(\lambda) = 0$ for $i \neq 2$, $b_4(0) = \pm 1$, $b_i(0) = 0$ for $i \neq 4$.

(iii) Use the theory of universal unfolding of singularities of functions to transform $a_0(\lambda)+a_1(\lambda)x+a_5(\lambda)x^5$ into $x^2+\varepsilon_1(\lambda)$.

iv) The last step consists in noting that for all monomials of the form $x^{3i+2}y$, the integrals $I_{3i+2}(h)$ (see Definition (2.4.1)) are, as functions of h, linearly dependent on the previous $I_k(h)$, $k<3i+2$. So, even if we keep the terms $x^{3i+2}y$ in the second equation, these terms do not contribute to the bifurcation diagram, which is studied through the zeros of the function $\bar{M}(h)$ [13]. (We learned recently that Baider and Sanders have been able to eliminate these terms, by a further reduction to normal form [3]).

Then we show that the bifurcation diagram of the system in reduced form is the same as that of (3.1) [13]. This is again studied through a change of coordinates like (3.3), but with $\varepsilon_1=\delta^4 v_0$.

The bifurcation diagram is given in a neighborhood of zero which is constructed as a union of cones:

(i) The half-space $\varepsilon_1>0$.

(ii) A cone K_4 constructed around the ε_4-axis on a *small* neighborhood in $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$-space.

(iii) A cone K_3 constructed around the ε_3-axis on the product of a *small* neighborhood in $(\varepsilon_1, \varepsilon_2)$-space with an *arbitrary* compact set in ε_4-space.

(iv) A cone K_2 constructed around the ε_2-axis on the product of a *small* neighborhood in ε_1-space with an *arbitrary* compact set in $(\varepsilon_3, \varepsilon_4)$-space.

(v) A cone K_1 constructed around the ε_1-axis on an *arbitrary* compact set in $(\varepsilon_2, \varepsilon_3, \varepsilon_4)$-space.

A proper choice of arbitrary compact sets will produce a neighborhood of the origin.

The last cone K_1 is the one obtained using the development of Section 2. For the other cones K_i we must use the universal unfolding of the cusps of order $i-1\leqslant 3$, the cusp of order 1 being simply the saddle-node [13].

APPENDIX: LYAPUNOV COEFFICIENTS METHOD

The Lyapunov coefficients method is one way to analyse Hopf bifurcation. We start with a system:

$$\begin{aligned} \dot{x} &= \lambda x - y + p(x,y) \qquad & p(x,y) &= O(|x,y|^2) \\ \dot{y} &= x + \lambda y + q(x,y) \qquad & q(x,y) &= O(|x,y|^2). \end{aligned} \tag{A.1}$$

When $\lambda = 0$, we look for a function

$$F(x, y) = (x^2 + y^2)/2 + O(|x, y|^3), \tag{A.2}$$

such that

$$\dot{F} = \sum V_i(x^2 + y^2)^{i+1}. \tag{A.3}$$

The coefficients V_i are called the *Lyapunov coefficients* of (A.1). These are obtained by making terms of the same degree on either side of the equality in (A.3) identical. The information contained in the Lyapunov coefficients is equivalent to the information given by the Hopf bifurcation coefficients in the following sense.

PROPOSITION [6]. *System* (A.1) *has a Hopf bifurcation of order k if and only if*

$$\lambda = V_1 = \cdots = V_{k-1} = 0, \qquad V_k \neq 0. \tag{A.4}$$

We can choose arbitrarily small perturbations of the system with k limit cycles, provided λ, $V_1, ..., V_k$ have alternating signs and

$$|\lambda| \ll |V_1| \ll \cdots \ll |V_k|. \tag{A.5}$$

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